Moment-angle complexes and polyhedral products for convex polytopes.

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ABSTRACT. Let P be a convex polytope not simple in general. In the focus of this paper lies a simplicial complex K_P which carries complete information about the combinatorial type of P. In the case when P is simple, K_P is the same as ∂P^*, where P^* is a polar dual polytope. Using the canonical embedding of a polytope P into nonnegative orthant $\mathbb{R}^m_{\geqslant}$, where m is a number of its facets, we introduce a moment-angle space $\mathcal{Z}_P$ for a polytope P. It is known, that in the case of a simple polytope P the space $\mathcal{Z}_P$ is homeomorphic to the moment-angle complex $(D^2, S^1)^{K_P}$. When P is not simple, we prove that the space $\mathcal{Z}_P$ is homotopically equivalent to the space $(D^2, S^1)^{K_P}$. This allows to introduce bigraded Betti numbers for any convex polytope. A Stanley-Reisner ring of a polytope P can be defined as a Stanley-Reisner ring of a simplicial complex K_P. All these considerations lead to a natural question: which simplicial complexes arise as K_P for some polytope P? We have proceeded in this direction by introducing a notion of a polytopic simplicial complex. It has the following property: link of each simplex in a polytopic complex is either contractible, or retractible to a subcomplex, homeomorphic to a sphere. The complex K_P is a polytopic simplicial complex for any polytope P. Links of so called face simplices in a polytopic complex are polytopic complexes as well. This fact is sufficient enough to connect face polynomial of a simplicial complex K_P to the face polynomial of a polytope P, giving a series of inequalities on certain combinatorial characteristics of P. Two of these inequalities are equalities for each P and represent Euler-Poincare formula and one of Bayer-Billera relations for flag f-numbers. In the case when P is simple all inequalities turn out to be classical Dehn-Sommerville relations.

1. Introduction

This work is devoted to the application of the theory of moment-angle complexes developed for simple polytopes and simplicial complexes to the study of nonsimple convex polytopes.

Let P be a convex polytope with m facets. There is a classical construction of convex geometry which allows to associate to each such polytope a dual polar polytope P^*. It can be defined geometrically, by using the construction of polar set in a euclidian space. The crucial property is the fact that the partially ordered set (poset) of faces of P^* coincides with the poset of faces of P with the reversed order (see details in [**20**]).

In the case when P is a simple polytope of dimension n (that is any vertex is contained in exactly n facets) the dual polytope P^* is simplicial (that is all its facets are simplices). So one can consider the boundary ∂P^* as a geometrical realization of an abstract simplicial complex which is often denoted K_P in a literature ([**9**],[**10**],[**16**]). The simplicial complex K_P

carries complete information about combinatorial type of a simple polytope P. Switching between a simple polytope P and simplicial complex K_P corresponding to this polytope allows to get different results in toric topology. We now briefly observe one example since it will make sense for the topic of our paper (all definitions will be given in the text).

The moment-angle complex $\mathcal{Z}_P$ of a simple polytope P can be defined in many different ways. There are two constructions of moment-angle complex which give homeomorphic topological spaces for a simple polytope P. One can use a geometrical embedding of P into the nonnegative orthant $\mathbb{R}^m_{\geqslant}$ to define $\mathcal{Z}_P$ as a pullback of the moment-angle map under such inclusion [**8**]. Such definition allows to define the structure of a smooth manifold on $\mathcal{Z}_P$. On the other hand, $\mathcal{Z}_P$ can be defined as a moment-angle complex $(D^2, S^1)^{K_P}$. Such a characterization appeared in [**8**], [**3**] and allowed to calculate cohomology ring of $\mathcal{Z}_P$. The term "polyhedral product" was used in [**2**], where the construction of a polyhedral product of the set of pairs $(X_i, A_i), i = 1, \ldots, m$ was considered and where the results were obtained about homotopical properties of these spaces. The space $(X, A)^K$ can be viewed as a particular case of a polyhedral product. We will use the term "polyhedral product" for spaces $(X, A)^K$ in the paper and the term "moment-angle complex" for its particular case $(D^2, S^1)^K$.

As we see, both definitions of a moment-angle complex of a simple polytope can be used for different purposes.

Note that for a simple polytope P the complex K_P can be defined without using dual polytope P^*. Indeed, let $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$ be facets of P. These facets cover the set ∂P. Taking the nerve of this cover, we get exactly K_P. Such observation can be taken as a definition of K_P.

For the sake of simplicity we do not distinguish between abstract simplicial complexes and their geometrical realizations. The simplicial complex hereafter means a finite combinatorial object as well as a topological space. When we write that some simplicial complex is homotopically equivalent to the space we mean that its geometrical realization is homotopically equivalent to this space. When we establish the equality of two complexes we mean that they are isomorphic as combinatorial objects. We hope that this agreement will not lead to misunderstanding.

In the case of general convex polytopes P the space ∂P^* is not a simplicial complex, therefore it does not coincide with K_P in any sense. So the study of K_P for nonsimple polytopes P differs from the study of dual polytopes. But even in this general case the complex K_P have many nice properties which we would like to discuss in this paper. An interesting thing is that the complex K_P carries a specific structure which we call *polytopic simplicial complex* (see definition 4.1).

Some of these properties generalize results about simple polytopes. We will try to underline connections with classical theory where it is possible. In particular, there had been found a formula for the f-polynomial of K_P which gives Dehn-Sommerville relations in the case when P is simple. The technique used to prove it is borrowed from theory of two-parametric face polynomials and the ring of polytopes, introduced in [**7**]. There is a key point in the original theory of ring of polytopes — the commutation property $F(dP) = \frac{\partial}{\partial t}F(P)$, which allows to express a face polynomial of P by face polynomials of its facets. This property allows to deduce Dehn-Sommerville relations from Euler-Poincare formula for a polytope. Big part of this theory can be generalized to deal with simplicial complex K_P. We introduce

two-dimensional face polynomial of a convex polytope P which is a combinatorial invariant of P. The coefficients of this polynomial are subject to a series of certain inequalities. Two of these inequalities turn out to be equalities for each convex polytope. One of them represents Euler-Poincare formula for a polytope, which is not very surprising. But another equality represents a certain relation on flag f-numbers of a polytope. The latter is one of Bayer-Billera relations [**5**].

The last part of the work is devoted to the properties of moment-angle spaces for general convex polytopes. As we already mentioned, the study of moment-angle complexes and polyhedral products of simplicial complexes is well developed. We tried to adapt this theory to deal with moment-angle spaces of nonsimple polytopes. Theorem 6.2 allows to switch between a moment-angle space of a polytope P and a moment-angle complex of a corresponding simplicial complex K_P even when P is not simple. This allows to introduce Betti numbers of a convex polytope.

We also underline that moment-angle complex is defined not only for simplicial complex but for any *hypergraph* also. It is natural to consider such moment-angle complexes when dealing with polytopes, though they give nothing new in a topological setting.

The structure of the paper is the following. In section 2 we define a characteristic hypergraph G_P of a polytope P and a simplicial complex K_P. These constructions carry complete information about combinatorial type of a polytope P. This means that a partially ordered set of facets of P can be uniquely determined by K_P or G_P. We also observe an operation of a simplicial closure of an arbitrary hypergraph. In terms of this operation K_P is a simplicial closure of G_P.

In section 3 we review the connection of K_P with the dual polytope P^*. It seems reasonable to consider K_P as some sort of simplicial resolution of P^*. We would like to underline the crucial idea: for nonsimple polytopes simplicial complex K_P does not coincide with ∂P^* in any sense.

There are connections of K_P with the structure of partially ordered set of faces of P. The complex K_P, as every simplicial complex, carries the structure of partially ordered set: its simplices are ordered by inclusion. Also the faces of P are partially ordered by inclusion. The connection between these posets is not the same as the connection between posets of faces of P and P^*. Indeed, the partially ordered set of faces of a polytope P^* is the same as the set of faces of a polytope P with reversed order. The situation in the case of K_P is different. In section 4 we observe some combinatorial properties of K_P. In particular, we describe how to restore a structure of P from the structure of K_P.

We have found one necessary condition for a simplicial complex to be K_P for some P. It is interesting that this condition leads to some new class of simplicial complexes. This is the class of so named *polytopic simplicial complexes*. The exact definition will be given in section 4, but, roughly speaking, the polytopic complex is a simplicial complex in which all links are either contractible or homotopically equivalent to a sphere. It means that polytopic complex looks rather like a manifold with a boundary but in homotopical sense. We prove that K_P is a polytopic complex for any convex polytope P and extract the information about f-vector of K_P using this structure.

The rest of the work concerns moment-angle complexes for different objects. Two types of moment-angle complexes are defined: a moment-angle space $\mathcal{Z}_P$ of a polytope and a moment-angle complex $\mathcal{Z}_G$ of a hypergraph. The first construction [**8**] uses the canonical

embedding of P into nonnegative orthant $\mathbb{R}^m_{\geqslant}$. The second construction [**8**] generalizes a notion of moment-angle complex $(D^2, S^1)^K$ for a simplicial complex K. In the case when P is simple, the moment-angle space of P is homeomorphic to a moment-angle complex of K_P as we already mentioned. But in general case these two spaces are not homeomorphic.

Nevertheless these spaces are very similar for different reasons. First of all, both $\mathcal{Z}_P$ and $\mathcal{Z}_{K_P}$ are equipped with an action of a torus. For any space X with an action of a torus one can consider an invariant $s(X)$ which is a maximal rank of toric subgroups acting freely on this space. In the case of moment-angle spaces of polytopes (or moment-angle complexes of simplicial complexes) arises an action of m-torus. Then the number s is a combinatorial invariant of a polytope (resp., simplicial complex). This invariant of polytopes and simplicial complexes was called the Buchstaber number and was considered in [**15**],[**11**],[**12**],[**1**]. We will show that $s(\mathcal{Z}_P) = s(\mathcal{Z}_{K_P})$ for any convex polytope P in section 5.

Another reason why K_P is a nice substitute of P is that $\mathcal{Z}_{K_P}$ is homotopically equivalent to $\mathcal{Z}_P$. We prove this fact in section 6 using two other definitions of $\mathcal{Z}_P$ which are equivalent to the original one. First definition describes $\mathcal{Z}_P$ as an identification space of $P \times T^m$ by some equivalence relation. This is similar to the situation with simple polytopes (see [**9**]). Another definition uses certain homotopy colimit [**17**].

Since $\mathcal{Z}_P \simeq \mathcal{Z}_{K_P}$ we may consider an additional grading in the ring $H^*(\mathcal{Z}_P, \mathbb{Z})$ as well as Betti numbers $\beta^{-i,2j}(P)$ of a polytope P. The Hochster formula

$$\beta^{-i,2j}(P) = \sum_{\omega \subseteq [m], |\omega| = j} \dim_{\mathbb{Z}} \tilde{H}^{j-i-1}(F_\omega; \mathbb{Z}),$$

where $F_\omega = \bigcup_{i \in \omega} \mathcal{F}_i \subseteq P$ is a union of facets, also makes sense in the case of non-simple polytopes. For the information on Betti numbers and moment-angle complexes see [**3**],[**9**],[**14**],[**16**].

So forth Betti numbers and cohomology of $\mathcal{Z}_P$ can be used as combinatorial invariants of a polytope P.

There are though important differences between moment-angle spaces for simple and nonsimple polytopes. For example, the restriction of the moment-angle map over the facet of a polytope cannot be described by a moment-angle complex over the facet as it is in the case of simple polytope. See details in example 6.1.

As a byproduct of our study we constructed some new invariants of convex polytopes. This may help in distinguishing combinatorial types of polytopes. There are many open problems in this area, for example:

Problem 1. *Let P be a convex polytope such that $P = P^*$. What can we say about the ring $H^*(\mathcal{Z}_{K_P})$ and invariant $s(P)$ for such P?*

The answer may help in understanding the properties of self-dual polytopes.

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2. Basic definitions and constructions

Let us fix a finite set $[m] = \{1, \ldots, m\}$. We will use a notion of a hypergraph in our paper. By definition, a hypergraph is an arbitrary system of subsets of $[m]$, $G \subseteq 2^{[m]}$. Elements of G are called hyperedges.

A simplicial complex is an important special case of a hypergraph. By definition, a simplicial complex is such a hypergraph K that: if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$.

It is usual in the literature to add a condition that all singletons lie in K. But we will not need this assumption.

Let G be a hypergraph. Let $K_G \subseteq 2^{[m]}$ be the minimal simplicial complex, such that $G \subseteq K_G$. Such a simplicial complex K_G is called a simplicial closure of G.

Other objects under consideration are polytopes. Let $P = \{x \in \mathbb{R}^n \mid \langle a_i, x\rangle + b_i \geqslant 0\}$ be a convex polytope, where $a_i \in \mathbb{R}^n,\ i = 1, \dots, m$ are the rows of $(m \times n)$-matrix A. We obtain the embedding

$$j_P \colon P \longrightarrow \mathbb{R}^m_{\geqslant} \, : \, j_P(x) = (y_1, \dots, y_m) \text{ where } y_i = \langle a_i, x\rangle + b_i.$$

Using j_P we will consider P as a polytope in $\mathbb{R}^m$.

For a convex polytope $P \subset \mathbb{R}^n$ its polar set $P^* \subset (\mathbb{R}^n)^*$ is defined as $P^* = \{l \in (\mathbb{R}^n)^* \mid \langle l, x\rangle \geqslant -1 \text{ for any } x \in P\}$. If $\dim P = n$ and the origin lies inside P, then P^* is a polytope. By polar (dual) polytope we mean P^* in this case. The partially ordered set of its faces coincides with the partially ordered set of faces of P with reversed order. See [**20**] for basic information on polytopes and polar duality.

An n-dimensional polytope P is called simple if any vertex of P is contained in exactly n facets. A polytope is called simplicial if its facets are simplices. The polar to a simple polytope is a simplicial polytope.

Take the function $\sigma \colon \mathbb{R}^m \to 2^{[m]} \, : \, \sigma(y) = \{i \in [m] \, : \, y_i = 0\}$. The image of σ is the set of all subsets of set $[m] = \{1, \dots, m\}$. We may restrict function σ to the image of a polytope in $\mathbb{R}^m$. This gives a function $\tilde{\sigma} = \sigma \circ j_p \colon P \to 2^{[m]}$. This function may be characterized another way. Let $\mathcal{F}_1, \dots, \mathcal{F}_m$ be facets of P given by equations $\mathcal{F}_i = \{x \in P \mid \langle a_i, x\rangle + b_i = 0\}$. Therefore we can identify elements of $[m]$ with facets of a polytope in an obvious way. Then for a point $x \in P$ we have $\tilde{\sigma}(x) = \{\imath \mid x \in \mathcal{F}_i\}$.

The image of a function $\tilde{\sigma}$ is a system of subsets of $[m]$, in other words, a hypergraph. We denote this hypergraph by G_P. Taking the simplicial closure of this hypergraph, we obtain a simplicial complex K_P. These two objects provide an important information about convex polytope.

Proposition 2.1. *The simplicial complex K_P can be characterized as follows. A subset $\{i_1, \dots, i_k\}$ is a simplex of K_P whenever $\mathcal{F}_{i_1} \cap \ldots \cap \mathcal{F}_{i_k} \neq \varnothing$ in a polytope P.*

The proof follows from definitions. In other words the simplicial complex K_P may be thought of as a nerve of the cover in sense of Pavel Alexandrov. The boundary of a polytope is covered by its facets. Then, by definition, the nerve of this cover is K_P. An important thing to note is the following: the dimension of K_P can be arbitrarily large even when the dimension n of P is fixed. In general, we have $n - 1 \leqslant \dim K_P \leqslant m - 2$ and this inequality cannot be improved. An example of a polytope P with $\dim K_P = m - 2$ is a pyramid (see ex. 5.1 in section 5). One can easily prove that $\dim K_P = n - 1$ if and only if a polytope P is simple.

Example 2.1. Let P be a pyramid over a square shown on figure 1 with a fixed enumeration of facets. The values of a function $\tilde{\sigma}$ are the following subsets: $\varnothing$ — is the image of $\tilde{\sigma}$ on the interior of P; $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$ — correspond to facets of P; $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 5\}$, $\{3, 5\}$, $\{4, 5\}$ — correspond to edges of the pyramid and

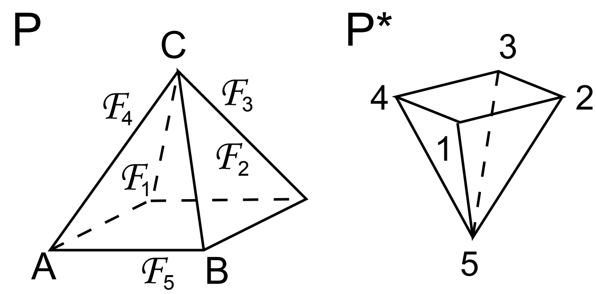

FIGURE 1. Pyramid over a square and its dual polytope

$\{1,2,5\}$, $\{2,3,5\}$, $\{3,4,5\}$, $\{1,4,5\}$, $\{1,2,3,4\}$ — these are the values $\tilde{\sigma}$ on vertices of P. We see that these subsets do not form a simplicial complex since $\{1,2,3,4\} \in G_P$, but $\{1,2,3\} \notin G_P$. But we may consider a simplicial closure K_P of G_P. Its maximal simplices are $\{1,2,5\}, \{2,3,5\}, \{3,4,5\}, \{1,4,5\}, \{1,2,3,4\}$. The complex K_P has dimension 3. It is not pure in this case.

The figure 1 also shows the polar polytope to a pyramid, which is actually a pyramid itself. We see that polar polytope is an object different from K_P in any sense.

REMARK. All the combinatorial structure of P can be recovered from G_P or K_P. Indeed, the set $G_P \subset 2^{[m]}$ is partially ordered by inclusion. This partially ordered set is isomorphic to the set of faces of P with reversed order, since any face determines a unique hyperedge of G_P. Also G_P is determined by K_P since every hyperedge in G_P is a multiple intersection of some maximal simplices in K_P. This fact will be verified in lemma 4.3. Connections of all the structures are described in section 4.

To get invariants of non-simple polytopes we need definitions for spaces associated with them. Let us consider a pullback of a diagram

$$\begin{tikzcd}
\mathcal{Z}_P \arrow[r, hook] \arrow[d] & \mathbb{C}^m \arrow[d, "p"'] \\
P \arrow[r, hook, "j_P"] & \mathbb{R}^m_{\geqslant}
\end{tikzcd}$$

The vertical map on the right is given by: $p\colon (z_1, \ldots, z_m) \mapsto (|z_1|^2, \ldots, |z_m|^2)$.

Like in the case of simple polytopes we call the space $\mathcal{Z}_P$ a moment-angle space of a polytope P. The space $\mathcal{Z}_P$ may be described as an intersection of real quadrics in $\mathbb{C}^m$ but in the case of non-simple polytope P this intersection has singularities. There is a canonical coordinatewise action of a torus T^m on $\mathcal{Z}_P$. The orbit space of this action is P itself.

REMARK. One would expect that the restriction of a moment-angle space to the facet gives a moment-angle space of this facet multiplied by complementary torus as it is in the case of simple polytope. This is not true in general. See example 6.1 in section 6.

For any hypergraph G and a pair of topological spaces (X, A) a polyhedral product

$$(X, A)^G = \bigcup_{\alpha \in G} X^\alpha \times A^{[m]\setminus\alpha}$$

is defined. This is a straightforward generalization of a polyhedral product for simplicial complexes. Indeed, it can be easily proved that $(X, A)^G = (X, A)^{K_G}$ where K_G is a simplicial closure of G.

As an example of a polyhedral product we would like to consider moment-angle complex of a hypergraph $\mathcal{Z}_G = (D^2, S^1)^G$. This definition coincides with a standard definition when a hypergraph is a simplicial complex. Also $\mathcal{Z}_G = \mathcal{Z}_{K_G}$ for any hypergraph G.

Another example of a polyhedral product is a complement to the coordinate space arrangement. Let us fix a hypergraph G. For each hyperedge $e \in G, e \subseteq [m]$ consider the plane $L_e = \{y \in \mathbb{C}^m \mid y_i = 0, i \notin e\}$. Then $L = \{L_{e_i} \mid e_i \in G\}$ is a coordinate space arrangement. Consider the complement to all spaces from the arrangement $U_G = \mathbb{C}^m \setminus \bigcup L_{e_i}$. In fact, we have $U_G = (\mathbb{C}, \mathbb{C}_*)^G$. As in the previous case, $U_G = U_{K_G}$.

Remark. The crucial point here is that we can define spaces U_G and $\mathcal{Z}_G$ not only for simplicial complexes but also for hypergraphs. But really we can get nothing new since $U_G = U_{K_G}$ and $\mathcal{Z}_G = \mathcal{Z}_{K_G}$.

A well known fact states that there is one-to-one correspondence between simplicial complexes on m vertices and spaces $U_K \subset \mathbb{C}^m$ which are complements to coordinate-space arrangements. In the case of hypergraphs a one-to-one correspondence between G and U_G does not exist.

Remark. Spaces $\mathcal{Z}_G$ and U_G are homotopically equivalent ([**9**], section 9).

To summarize all these definitions consider a scheme

$$\begin{tikzcd}
P \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dd, mapsto, bend right] & \mathcal{Z}_P & \\
G_P \arrow[r, mapsto] \arrow[d, mapsto] \arrow[rr, bend left] & \mathcal{Z}_{G_P} \arrow[r, hook, "\simeq"'] \arrow[d, equal] & U_{G_P} \arrow[d, equal] \\
K_P \arrow[r, mapsto] \arrow[rr, bend right] & \mathcal{Z}_{K_P} \arrow[r, hook, "\simeq"] & U_{K_P}
\end{tikzcd} \tag{1}$$

We have described two topological constructions: a moment-angle space for a polytope and a moment-angle complex of a hypergraph. The spaces $\mathcal{Z}_P$ and $\mathcal{Z}_{K_P}$ does not coincide in general. We discuss the connection of these spaces in section 6.

Let us review constructions of this section in the case of simple polytopes.

Proposition 2.2. *Let P be a simple polytope. Then $G_P = K_P$ and the geometric realization of K_P coincides with ∂P^*. Also $\mathcal{Z}_P \cong \mathcal{Z}_{K_P}$.*

Proof. The proof of the first statement easily follows from properties of simple polytopes ([**20**]). The last statement is a standard fact in toric topology [**9**]. $\square$

3. Interconnections between polytopes and simplicial complexes

Let P be a convex polytope. A scheme shows how simplicial complex K_P is connected to the polar polytope P^*.

$$\begin{tikzcd} P \arrow[rr, mapsto] \arrow[dr, mapsto] & & K_P \\ & P^* \arrow[ur, mapsto] & \end{tikzcd}$$

Here:

(1) $P \mapsto P^*$ is an association of a polar dual polytope.

(2) For a given polytope Q one can build a simplicial complex $\hat{Q}$. Vertices of $\hat{Q}$ are vertices of Q. A set of vertices forms a simplex of $\hat{Q}$ whenever these vertices are contained in a common facet of Q. In the case when Q is simplicial there holds $\partial Q = \hat{Q}$.

In the terms introduced above $\widehat{(P^*)} = K_P$. If P is simple, then $K_P = \widehat{(P^*)} = \partial P^*$.

There is a canonical piecewise linear map from $\hat{Q}$ to ∂Q (here we do not distinguish between abstract simplicial complex $\hat{P}$ and its geometrical realization). The map $p\colon \hat{Q} \to \partial Q$ is defined on vertices as identity map. Then it can be continued linearly to each maximal simplex of $\hat{Q}$ thus giving a piecewise linear map $p\colon \hat{Q} \to \partial Q$. Such a map is well defined since an image of any maximal simplex of $\hat{Q}$ is a facet of Q.

Proposition 3.1. *Let Q be a polytope of dimension n. There exists a piecewise linear embedding $i\colon \partial Q \hookrightarrow \hat{Q}$ such that $p \circ i = id_{\partial Q}$ and the image $Z = i(\partial Q)$ is a simplicial subcomplex of $\hat{Q}$ satisfying the properties:*

1) Z is homeomorphic to a sphere S^{n-1};

2) Z is a strong deformation retract of $\hat{Q}$.

Proof. Let us fix a canonical geometrical realization of a complex $\hat{Q}$ (that is a realization in Euclidian space such that all simplices are convex).

Suppose that there is a triangulation of a polytope Q by convex simplices such that vertices of any simplex from this triangulation are vertices of Q. If Q is triangulated as described, we consider an induced triangulation Δ of the boundary ∂Q. Once the triangulation Δ is fixed, we define a map $i\colon \partial Q \to \hat{Q}$. The map i is defined on vertices as identity (vertices of Q are, by definition, in 1-to-1 correspondence with vertices of $\hat{Q}$). The map i can be extended by linearity to each simplex of triangulation Δ. This is well defined map to the complex $\hat{Q}$ since any simplex σ from the triangulation Δ is contained in some facet of Q thus its image lies in some simplex of $\hat{Q}$. Obviously, i is an embedding and $Z = i(\partial Q)$ is a simplicial subcomplex of $\hat{Q}$. Also, Z is homeomorphic to ∂Q which is homeomorphic to a sphere S^{n-1}. By linearity and all the definitions, we have $p \circ i = id_{\partial Q}$.

For a point $x \in \partial Q$ the preimage $p^{-1}(x)$ is a convex subset of $\hat{Q} \subset \mathbb{R}^m$. This preimage contains $i(x)$ as we have already seen. The required strong retraction of $\hat{Q}$ on $i(\partial Q)$ can be easily constructed using linear contraction of each fiber $p^{-1}(x)$ to a point $i(x)$.

We have proved the proposition under the assumption that any polytope Q can be triangulated by convex simplices in such a way that vertices of a triangulation are those

of Q. To find such a triangulation we use a technique of Delaunay triangulation on the set of vertices of Q. If Q is a simplex we have nothing to prove. Otherwise, without loss of generality assume that vertices of Q do not lie on the same sphere S^{n-1} (if they do, apply an affine transformation to a polytope Q).

We introduce an additional dimension to the linear span of the polytope so that $Q \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$. Consider a paraboloid in $\mathbb{R}^{n+1}$ defined by the equation $z = x_1^2 + \ldots + x_n^2$, where coordinate z corresponds to the additional dimension. Lift all the vertices of polytope Q to the paraboloid. More strict: consider a set of points $A_i = (v_{1i}, \ldots, v_{ni}, v_{1i}^2 + \ldots + v_{ni}^2)$ for all vertices $v_i = (v_{i1}, \ldots, v_{in})$ of Q. Now span a convex hull H on points A_i in the space $\mathbb{R}^{n+1}$ and take its lower part L (this is a standard technique when dealing with Delaunay triangulations, see [**13**] for information). The lower part L is subdivided to convex polytopes. If we take a projection of L back to the subspace $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ we get a subdivision of Q by convex polytopes R_j; each of them is spanned by some vertices of Q. Moreover, all R_j have less vertices than Q. Indeed, if Q is the only lower face of $H \subset \mathbb{R}^{n+1}$, then all the points A_i lie in the same hyperplane. Therefore, all vertices of the polytope Q lie on a sphere (see [**13**]) which contradicts the assumption.

We can construct a triangulation of each R_j by induction on the number of vertices. The proof is now completed. $\square$

The construction of $\hat{Q}$ can be viewed as simplicial resolution of a polytope Q. We suggest that it can be used in studying singular vertices of nonsimple polytopes.

Proposition 3.2. *Let P and R be convex polytopes. Then $K_{P\times R} = K_P * K_R$ and $\widehat{P \circ R} = \hat{P} * \hat{R}$, where $\times$ is a direct product of polytopes, $\circ$ is a convex product of polytopes and $*$ is a join of simplicial complexes (see* [**9**] *for definitions of these operations).*

Proof. Denote facets of P by $\mathcal{F}_1, \ldots, \mathcal{F}_m$ and facets of R by $\mathcal{F}'_1, \ldots, \mathcal{F}'_l$. Then facets of $P \times R$ are $\mathcal{F}_1 \times R, \ldots, \mathcal{F}_m \times R, P \times \mathcal{F}'_1, \ldots, P \times \mathcal{F}'_l$. Then $\mathcal{F}_{i_1} \times R \cap \ldots \cap \mathcal{F}_{i_k} \times R \cap P \times \mathcal{F}'_{j_1} \cap \ldots \cap P \times \mathcal{F}'_{j'_l} \neq \varnothing$ if and only if $\mathcal{F}_{i_1} \cap \ldots \cap \mathcal{F}_{i_k} \neq \varnothing$ and $\mathcal{F}'_{j_1} \cap \ldots \cap \mathcal{F}'_{j'_l} \neq \varnothing$. This means that $K_{P\times R} = K_P * K_R$.

Second fact follows from the series of identities $\widehat{P \circ R} = (\widehat{P^* \times R^*})^* = K_{P^* \times R^*} = K_{P^*} * K_{R^*} = \hat{P} * \hat{R}$. $\square$

Remark. Combinatorial invariants of simplicial complexes K_P and $\widehat{P}$ are the invariants of a polytope P. For example, face vector of K_P is an invariant of P (see section 4). In sections 5 and 6 Buchstaber number and Betti numbers of K_P are considered.

4. The structure of K_P

The key question for us is the following: given an arbitrary simplicial complex K, is there a convex polytope P such that $K = K_P$?

The complete answer to this question seems to be very difficult and unattainable ([**4**],[**9**]). Indeed, the problem being restricted to the class of simple polytopes has the form: which simplicial complexes arise as boundaries of simplicial polytopes? Any complex of that kind is a simplicial sphere, but not all the spheres are boundaries of simplicial polytopes. The Barnette sphere [**4**] is not a boundary of any convex simplicial polytope. See other examples and references in [**9**].

Here we discuss the general situation: what are the necessary conditions for a simplicial complex to be equivalent to a complex K_P for some convex polytope P. We found such a condition and as a byproduct get the formula for the f-vector of a simplicial complex K_P.

We will need a small technical lemma to work with K_P.

Lemma 4.1. *Let $\mathcal{C}_2 \subseteq \mathcal{C}_1$ be two finite contractible covers of the same space. Then nerve $\mathcal{N}_2$ of the cover $\mathcal{C}_2$ is a strong deformation retract of the nerve $\mathcal{N}_1$ of the cover $\mathcal{C}_1$.*

Proof. Without loss of generality it can be assumed that the cover $\mathcal{C}_2 = \{A_1, \dots, A_k\}$ and $\mathcal{C}_1 = \mathcal{C}_2 \cup \{A_{k+1}\}$, that is $\mathcal{C}_1$ has only one extra element which is not contained in $\mathcal{C}_2$. Then, obviously, $\mathcal{N}_2 \subseteq \mathcal{N}_1$. Moreover, we may write $\mathcal{N}_1 = \mathcal{N}_2 \cup_{\operatorname{link} v_{k+1}} \operatorname{star} v_{k+1}$, where v_{k+1} is a vertex of $\mathcal{N}_1$ which corresponds to a set A_{k+1}, and link and star are taken in the complex $\mathcal{N}_1$. By all definitions, $\operatorname{link}_{\mathcal{N}_1} v_{k+1}$ is a nerve of a cover of A_{k+1} by the sets $B_i = A_i \cap A_{k+1}$. Since all the sets and their intersections are contractible we found that $\operatorname{link}_{\mathcal{N}_1} v_{k+1}$ is contractible. But $\mathcal{N}_1 = \mathcal{N}_2 \cup_{\operatorname{link} v_{k+1}} \operatorname{star} v_{k+1} = \mathcal{N}_2 \cup_{\operatorname{link} v_{k+1}} \operatorname{cone}(\operatorname{link} v_{k+1}) = \mathcal{N}_2 \cup_Y \operatorname{cone}(Y)$, where Y is contractible. By standard reasoning, $\mathcal{N}_2$ is a strong deformation retract of $\mathcal{N}_1$. $\square$

Let P be a convex polytope and $\mathcal{F}_i$ — its facets. Denote the poset of all faces of a polytope P by $\mathcal{P}$. We now define a map $\tilde{\sigma}$ from $\mathcal{P}$ to K_P by the formula $\tilde{\sigma}(F) = \{i \mid F \subseteq \mathcal{F}_i\}$. This map was already defined in section 2 (but we considered it as a map defined on the points of P). The map $\tilde{\sigma}$ is injective and its image is exactly the hypergraph G_P by definition. Simplices from the image of $\tilde{\sigma}$ have very important property.

Proposition 4.2. *Let F be a face of a polytope P. Consider a simplex $\tilde{\sigma}(F)$ and its link $L_{\tilde{\sigma}(F)} = \operatorname{link} \tilde{\sigma}(F)$ in a simplicial complex K_P. In this case there is a simplicial subcomplex $Z_{\tilde{\sigma}(F)} \subseteq L_{\tilde{\sigma}(F)}$ such that:*

1) $Z_{\tilde{\sigma}(F)}$ is a strong deformation retract of $L_{\tilde{\sigma}(F)}$;

2) $Z_{\tilde{\sigma}(F)}$ is homeomorphic to a sphere $S^{\dim F - 1}$.

Proof. First of all, we prove that $L_{\tilde{\sigma}(F)}$ is homotopically equivalent to a sphere $S^{\dim F-1}$.

We have $F = \bigcap_{i \in \tilde{\sigma}(F)} \mathcal{F}_i$ and, moreover, $F \nsubseteq \mathcal{F}_j$ if $j \notin \tilde{\sigma}(F)$ by definition of function $\tilde{\sigma}$. The boundary ∂F is covered by the sets $G_j = \mathcal{F}_j \cap F = \mathcal{F}_j \cap \left(\bigcap_{i \in \tilde{\sigma}(F)} \mathcal{F}_i\right)$ because any point of the boundary ∂F is contained in some facet which does not contain F (these sets does not cover F itself by the reasoning above). We claim that the nerve of this cover is exactly the complex $\operatorname{link} \tilde{\sigma}(F)$. To prove this consider the series of equivalent statements: $G_{j_1} \cap \ldots \cap G_{j_s} \neq \varnothing \Leftrightarrow \mathcal{F}_{j_1} \cap \dots \cap \mathcal{F}_{j_s} \cap \left(\bigcap_{i \in \tilde{\sigma}(F)} \mathcal{F}_i\right) \Leftrightarrow \{j_1, \dots, j_s\} \sqcup \tilde{\sigma}(F) \in K_P \Leftrightarrow \{j_1, \dots, j_s\} \in L_{\tilde{\sigma}(F)}$. All sets G_j are convex so the cover of ∂F by G_i is contractible. Therefore $L_{\tilde{\sigma}(F)} \simeq \partial F \simeq S^{\dim F-1}$.

Now we construct a strong deformation retract $Z_{\tilde{\sigma}(F)}$ of $L_{\tilde{\sigma}(F)}$ which is homeomorphic to a sphere $S^{\dim F-1}$. Note that a face F is itself a polytope. Thus the complex K_F is defined. We will follow the plan: 1) prove that K_F is a strong deformation retract of $L_{\tilde{\sigma}(F)}$ 2) prove that there is a simplicial subcomplex $Z_{\tilde{\sigma}(F)} \subseteq K_F$ which is a strong deformation retract of K_F and which is homeomorphic to a sphere $S^{\dim F-1}$.

1) There are two contractible covers of a set ∂F. First cover $\mathcal{A} = \{G_j\}$ was defined earlier in the proof. Second cover is the cover $\mathcal{B} = \{H_t\}$ of the boundary ∂F by the facets of F. The nerve of $\mathcal{A}$ is a complex $\operatorname{link} \tilde{\sigma}(F)$ as was previously proved. The nerve of $\mathcal{B}$ is K_F by definition. It can be easily seen that each facet H_t of the cover $\mathcal{B}$ is contained in the cover $\mathcal{A}$. Therefore $\mathcal{B}$ is a subcover of $\mathcal{A}$.

The lemma 4.1 being applied to covers $\mathcal{B} \subseteq \mathcal{A}$ states that K_F is a strong deformation retract of $\operatorname{link} \tilde{\sigma}(F)$. It is obviously a simplicial subcomplex.

2) We should now prove that K_F has a subcomplex $Z_{\tilde{\sigma}(F)}$ which is a strong deformation retract of K_F and is homeomorphic to a sphere. By proposition 3.1, the boundary of the dual polytope ∂F^* can be embedded in K_F as a strong deformation retract. $\square$

The proposition 4.2 states that some of the simplices of K_P have very specific links. This condition is axiomatized in the following construction.

Let K be an arbitrary simplicial complex. Its maximal under inclusion simplices form a set $M(K) \subset K$. We call a simplex τ face simplex if it can be represented as an intersection of some number of maximal simplices $\tau = \sigma_1 \cap \ldots \cap \sigma_s$, $\sigma_i \in M(K)$. The set of all face simplices will be denoted by $F(K)$. Therefore $M(K) \subseteq F(K) \subseteq K$. The set $F(K)$ is naturally ordered by inclusion.

Definition 4.1. *Simplicial complex K is called polytopic complex (P-complex) of rank n if the following conditions hold:*

1) $\varnothing \in F(K)$, i.e. intersection of all maximal simplices of K is empty;

2) $F(K)$ is a graded poset of rank n (it means that all its maximal under inclusion chains have the same length $n+1$). In this case the rank function $\operatorname{rk}(\sigma)$ is defined. It satisfies the property $\operatorname{rk}(\varnothing) = 0$, and $\operatorname{rk}(\sigma) = n$ for any maximal simplex σ;

3) Suppose $\sigma \in F(K)$. Then there is a simplicial subcomplex Z_σ in simplicial complex $\operatorname{link}_K \sigma$ such that Z_σ is homeomorphic to a sphere $S^{n-\operatorname{rk}(\sigma)-1}$ and Z_σ is a strong deformation retract of $\operatorname{link}_K \sigma$. Here, by definition, $\operatorname{link} \varnothing = K$ and $S^{-1} = \varnothing$.

The P-complex K is called reduced if any singleton is a face simplex: $\{v\} \in F(K)$.

Remark. The rank function on a poset has the property $\operatorname{rk}(\sigma) = \operatorname{rk}(\tau) - 1$ if $\sigma \subset \tau$ and there is no element ρ such that $\sigma \subset \rho \subset \tau$. Consult [**18**] for details.

Remark. It follows from definition that in a P-complex link of any face simplex is homotopically equivalent to a sphere.

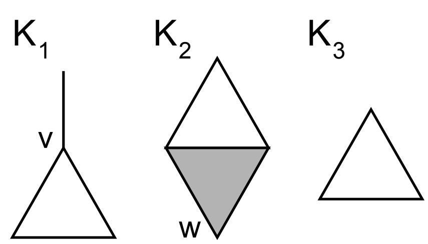

Figure 2.

Example 4.1. Figure 2 illustrates three simplicial complexes. The complex K_1 is not a P-complex. Indeed, v is a face simplex (it can be represented as an intersection of maximal simplices), but $\operatorname{link} v$ is a disjoint union of three points. Therefore $\operatorname{link} v$ is not homotopically equivalent to a sphere of any dimension. This violates the third condition in definition 4.1.

The complex K_2 is P-complex of rank 2. It is unreduced since the vertex $\{w\}$ is not an intersection of maximal simplices. The complex K_3 is an example of reduced P-complex of rank 2.

Lemma 4.3. *For any n-dimensional polytope P with m facets the complex K_P is a reduced P-complex of rank n. Moreover, $F(K_P) = G_P$ as a subset of $2^{[m]}$ and $F(K_P)$ is equivalent as a graded poset to the set of faces of P ordered by reverse inclusion.*

PROOF. At first, we will prove that $F(K_P)$ is equal to $G_P = \tilde{\sigma}(\mathcal{P})$.

It can be seen that vertices of P correspond to maximal simplices of K_P, since they are contained in the most number of facets. Now recall the basic fact: the face F is contained in the facet $\mathcal{F}$ iff all vertices of F are contained in $\mathcal{F}$. Therefore $\{i \mid F \subseteq \mathcal{F}_i\} = \bigcup_{v \in F} \{i \mid v \subseteq \mathcal{F}_i\}$. This expression shows that every element from the image of $\tilde{\sigma}$ is an intersection of some maximal simplices. Therefore $G_P = \tilde{\sigma}(\mathcal{P}) \subseteq F(K_P)$.

To prove that $\tilde{\sigma}$ is surjective map to $F(K_P)$ consider an arbitrary face simplex σ of K_P. By definition, $\sigma = \tau_1 \cup \ldots \cup \tau_s$, where each τ_j is maximal. Each τ_j have the form $\tau_j = \{i \mid v_j \subseteq \mathcal{F}_i\}$ for some vertex v_j of a polytope P. Summarizing, we have $\sigma = \{i \mid \mathcal{F}_i$ contains all v_j for $j = 1, \ldots, s\}$. Let F be the minimal face of P containing all vertices v_j. Then, obviously $\tilde{\sigma}(F) = \sigma$.

So forth there is a bijection between sets $\mathcal{P}$, G_P and $F(K_P)$ which respects the order. So the poset $F(K_P)$ is an ordered poset and one can define the rank function on it by $\mathrm{rk}(\sigma) = n - \dim(\tilde{\sigma}^{-1}(\sigma))$. We have $\mathrm{rk}(\varnothing) = 0$ since $\tilde{\sigma}^{-1}(\varnothing) = P$ and $\dim P = n$. Also $\mathrm{rk}(\sigma) = n$ when σ is maximal simplex since $\tilde{\sigma}^{-1}(\sigma)$ is vertex in this case.

It is clear that any vertex of K_P is a face simplex. Indeed, each vertex i of K_P corresponds to the facet $\mathcal{F}_i$ thus $\tilde{\sigma}(\mathcal{F}_i) = \{i\} \subseteq F(K_P)$. The empty set is an intersection of all vertices, so the first condition also holds.

The last thing we need to prove is property 3 in definition 4.1 which concerns links of face simplices. But this is exactly the statement of proposition 4.2. This completes the proof of the theorem. □

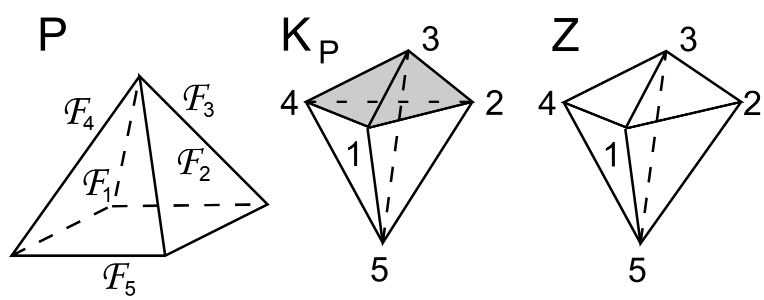

FIGURE 3.

EXAMPLE 4.2. Consider a pyramid P over a square as in example 2.1. Figure 3 shows the polytope P, simplicial complex K_P and the subcomplex $Z \subset K_P$. The complex $Z = Z_\varnothing = Z_{\tilde{\sigma}(P)}$ is composed of maximal simplices $\{1,3,4\}, \{1,4,5\}, \{3,4,5\}, \{1,2,3\}, \{1,2,5\}, \{2,3,5\}$. All of them are the simplices of K_P. The complex Z is homeomorphic to the sphere S^2. It is a retract of K_P as well.

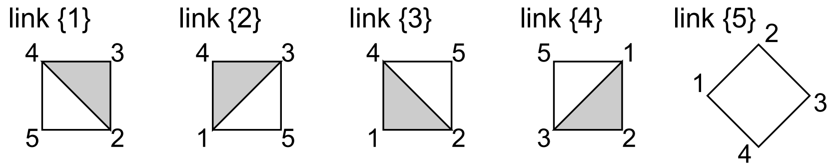

FIGURE 4.

Now we check that K_P is a P-complex. This is the list of all face simplices of K_P: maximal simplices $\{1,2,3,4\}, \{1,2,5\}, \{2,3,5\}, \{3,4,5\}, \{1,4,5\}$; double intersections of maximal simplices $\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}$ (all of them correspond to edges of P as they should); the vertices $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$ and the empty set $\varnothing$. Links of the vertices are depicted in figure 4. Note that all of them contain a circle S^1 as a retract. We do not illustrate links of all other face simplices. But one can easily check the property 3 from the definition of P-complex. For example, $\mathrm{link}\{1,2\}$ is a disjoint union of a point $\{5\}$ and an interval $\{3,4\}$, thus contains S^0 as a retract.

But if we look at the simplex $\{1,3\}$ which is not a face simplex, we see that $\mathrm{link}\{1,3\}$ is an interval, that is a contractible space. The situation goes the same with all other nonface simplices.

EXAMPLE 4.3. Any $(n-1)$-dimensional PL-sphere K is an P-complex of rank n and all its simplices are face simplices, so $F(K) = K$. Indeed, any $(n-2)$-dimensional simplex of K is contained in exactly two maximal simplices thus it is a face simplex. Suppose $\sigma \in K$ is an arbitrary simplex contained in some maximal simplex τ. Then σ can be represented as an intersection of $n-2$-dimensional subsimplices of τ, thus σ is a face simplex. The rank function is given by $\mathrm{rk}\,\sigma = \dim\sigma + 1$. The third condition of definition 4.1 holds by definition of PL-sphere.

EXAMPLE 4.4. Any P-complex K of rank 1 is a disjoint union of two simplices. Indeed, its maximal simplices should not intersect by definition. On the other hand, K should be homotopically equivalent to a pair of points.

Proposition 4.4. *If K is a P-complex of rank n and σ its face simplex of rank k, then $\mathrm{link}\,\sigma$ is a P-complex of rank $n-k$.*

PROOF. To prove this, consider a maximal simplex τ of the complex $\mathrm{link}\,\sigma$. Then $\sigma\sqcup\tau$ is the maximal simplex of K. And vice a versa: all maximal simplices of K containing σ have the form $\sigma\sqcup\tau$ where τ is the maximal simplex of $\mathrm{link}\,\sigma$. So $\tau \in M(\mathrm{link}\,\sigma) \Leftrightarrow \tau\sqcup\sigma \in M(K)$. The same obviously holds for face simplices: $\tau \in F(\mathrm{link}\,\sigma) \Leftrightarrow \tau\sqcup\sigma \in F(K)$. Therefore $F(\mathrm{link}\,\sigma)$ is isomorphic as a poset to the filter $F(K)_{\geqslant} = \{\tau \in F(K) \mid \tau \supseteq \sigma\}$. So the poset $F(\mathrm{link}\,\sigma)$ is a graded poset of rank $n-k$ and the rank function is defined by $\mathrm{rk}_{\mathrm{link}\,\sigma}\tau = \mathrm{rk}_K(\tau\sqcup\sigma) - k$ (there stands rank function on $F(K)$at the right). The third condition in definition 4.1 follows from identities: $\mathrm{link}_{\mathrm{link}\,\sigma}\tau = \mathrm{link}(\sigma\sqcup\tau) \simeq S^{n-\mathrm{rk}_K(\sigma\sqcup\tau)-1} = S^{n-k-\mathrm{rk}_{\mathrm{link}\,\sigma}\tau-1}$. $\square$

EXAMPLE 4.5. The only reduced P-complexes of rank 2 are the boundaries of polygons. Let K be a reduced P-complex of rank 2. Consider two sets: $M(K)$ — the set of maximal simplices of K and $V(K)$ — the set of vertices. Each vertex is a face simplex and has rank

1. Thus the link of any vertex is a disjoint union of two simplices by proposition 4.4 and example 4.4. Therefore any vertex is contained in exactly two maximal simplices. Consider a graph Γ on a set $M(K)$: two nods σ_1 and σ_2 are connected by an edge iff corresponding maximal simplices σ_1 and σ_2 share a common vertex. This graph is homotopically equivalent to K. One can prove this by considering the contractible cover of geometrical realization of K by maximal simplices. So forth, Γ is homotopically equivalent to a 1-sphere. Therefore it has the same number of edges and nodes. This means that $\#V(K) = \#M(K)$. Each maximal simplex has at least two vertices. Counting the number of pairs $v \subset m$, where $v \in V(K)$ and $m \in M(K)$, we find: $2\#V(K) = \#\{v \subset m\} \geqslant 2\#M(K)$. Since $\#V(K) = \#M(K)$ we find out that any maximal simplex have exactly two vertices, providing K to be a graph itself. Since the degree of any node of K is 2 and K is homotopically equivalent to a circle we conclude that K is a simple cycle, i.e. a boundary of a polygon.

Let τ be a simplex of K. If $\sigma_1, \sigma_2 \in F(K)$ and $\tau \in \sigma_1$, $\tau \in \sigma_2$, then $\sigma_1 \cap \sigma_2 \in F(K)$ and $\tau \in \sigma_1 \cap \sigma_2$. Therefore for every $\tau \in K$ there exists a minimal $\sigma \in F(K)$ containing τ (it is an intersection of all face simplices containing τ). We denote such face simplex by $\hat{\tau}$.

Lemma 4.5. *For any $\sigma \in K$ we have* $\operatorname{link}\sigma = \operatorname{link}\hat{\sigma} * (\hat{\sigma} \setminus \sigma)$.

PROOF. Any maximal simplex of K containing σ contains $\hat{\sigma}$ as well. Let τ be the maximal simplex of $\operatorname{link}\sigma$. Then $\sigma \sqcup \tau$ is a maximal simplex of K thus $\hat{\sigma} \setminus \sigma \subseteq \tau$. Therefore any maximal simplex τ of $\operatorname{link}\sigma$ can be written as $\tau = (\hat{\sigma} \setminus \sigma) \sqcup \check{\tau}$, where $\check{\tau}$ is the maximal simplex of $\operatorname{link}\hat{\sigma}$. This yields $\operatorname{link}\sigma = \operatorname{link}\hat{\sigma} * (\hat{\sigma} \setminus \sigma)$. $\square$

Corollary 4.6. *If σ is not a face simplex in complex K, then* $\operatorname{link}\sigma$ *is contractible.*

We see that in a P-complex links of all simplices are either contractible or homotopically equivalent to a sphere. To make use of this fact we observe some constructions which were used in the theory of the ring of polytopes [**7**]. This combinatorial technique was introduced to effectively count the number of faces of simple polytopes, in particular, to prove the Dehn-Sommerville relations. The same technique works in our situation as well.

Let K be an arbitrary simplicial complex. Consider an f-polynomial

$$f_K(t) = \sum_{\sigma \in K} t^{|\sigma|} = f_0 + f_1 t + f_2 t^2 + \ldots + f_n t^n + \ldots, \tag{2}$$

where f_i is the number of $(i-1)$-dimensional simplices of K, $f_0 = 1$.

Lemma 4.7. *For any simplicial complex K we have*

$$\frac{d}{dt} f_K(t) = \sum_{v \in V(K)} f_{\operatorname{link} v}(t). \tag{3}$$

PROOF. The proof consists in the computation:

$$\sum_{v\in K} f_{\operatorname{link} v}(t) = \sum_{v\in K}\sum_{\sigma\in\operatorname{link} v} t^{|\sigma|} =$$
$$= \sum_{\substack{v\in K,\sigma\in K,\\ v\notin\sigma, v\cup\sigma\in K}} t^{|\sigma|} = \sum_{\substack{\tau\in K, v\in\tau,\\ \sigma=\tau\setminus v}} t^{|\sigma|} = \sum_{\tau\in K, v\in\tau} t^{|\tau|-1} =$$
$$= \sum_{\tau\in K} |\tau| t^{|\tau|-1} = \frac{\partial}{\partial t} f_K(t).$$

□

Lemma 4.8. *For any simplicial complex K and for any natural s we have*

$$\left(\frac{d}{dt}\right)^s f_K(t) = s! \sum_{\sigma\in K, |\sigma|=s} f_{\operatorname{link}\sigma}(t).$$

PROOF. The proof uses an induction on s. The base of induction $s=1$ was proved in lemma 4.7. By induction hypothesis $\left(\frac{d}{dt}\right)^{s-1} f_K(t) = (s-1)! \sum\limits_{\sigma\in K,|\sigma|=s-1} f_{\operatorname{link}\sigma}(t)$. Differentiating this equality by t and using lemma 4.7, we get

$$\left(\frac{d}{dt}\right)^s f_K(t) = (s-1)! \sum_{\sigma\in K,|\sigma|=s-1}\sum_{v\in\operatorname{link}\sigma} f_{\operatorname{link}_{\operatorname{link}\sigma} v}(t) =$$
$$= (s-1)! \sum_{\sigma\in K,|\sigma|=s-1}\sum_{v\in\operatorname{link}\sigma} f_{\operatorname{link}(\sigma\sqcup v)}(t) = (s-1)! \sum_{\tau\in K,|\tau|=s}\sum_{v\in\tau} f_{\operatorname{link}\tau}(t) =$$
$$= s! \sum_{\tau\in K,|\tau|=s} f_{\operatorname{link}\tau}(t),$$

which was to be proved. □

Theorem 4.9. *For any P-complex K of rank n we have*

$$f_K(t) = \sum_{\sigma\in F(K)} (-1)^{n-\operatorname{rk}\sigma}(t+1)^{|\sigma|}. \tag{4}$$

PROOF. First of all, note that for any simplicial complex L there holds $f_L(-1) = 1-\chi(L)$, where $\chi(L)$ is an Euler characteristic of a simplicial complex. In particular,

$$f_L(-1) = \begin{cases} 0, \text{ if } L\simeq \mathrm{pt}; \\ (-1)^{l+1}, \text{ if } L\simeq S^l. \end{cases}$$

Now consider a Taylor series for the polynomial $f_K(t)$ at a point -1:

$$f_K(t) = f_K(-1) + \frac{1}{1!}\frac{df_K}{dt}(-1)(t+1) + \ldots + \frac{1}{s!}\frac{d^s f_K(-1)}{dt^s}(t+1)^s + \ldots. \tag{5}$$

Using lemma 4.8 and the observation about Euler characteristic, we have

$$\frac{1}{s!}\frac{d^s f_K(-1)}{dt^s} = \sum_{\sigma\in K,|\sigma|=s} f_{\operatorname{link}\sigma}(-1) = \sum_{\sigma\in F(K),|\sigma|=s} (-1)^{n-\operatorname{rk}\sigma}.$$

So the formula 5 may be extended:

$$f_K(t) = \sum_{s=0}^{\infty} \left[\sum_{\sigma \in F(K), |\sigma|=s} (-1)^{n-\mathrm{rk}\,\sigma} \right] (t+1)^s = \sum_{\sigma \in F(K)} (-1)^{n-\mathrm{rk}\,\sigma}(t+1)^{|\sigma|},$$

which gives the required result. □

Let P be a convex polytope and F — its face. Let $m(F)$ be a number of facets containing F.

Definition 4.2. *Two-dimensional face polynomial of a polytope P is the polynomial in two variables*

$$F_P(\alpha, t) = \sum_{F \subseteq P} \alpha^{\dim F} t^{m(F)}, \tag{6}$$

where the sum is taken over all faces of P including P itself. It is clear that two-dimensional polynomial is a combinatorial invariant of P

From theorem 4.9 and lemma 4.3 follows

Corollary 4.10. *For any convex polytope P there holds $f_{K_P}(t) = F_P(-1, t+1)$.*

We now observe briefly some basic facts from the theory of ring of polytopes developed in [**7**]. We would like to get some results in this theory as a particular cases of general considerations made above.

Let $\mathbb{P}_n$ be a free abelian group formally generated by combinatorial simple polytopes of dimension n (that is a group of all finite formal sums $\alpha_1 P_1 + \ldots + \alpha_s P_s$, where $\alpha_i \in \mathbb{Z}$ and P_s are simple combinatorial polytopes). Taking direct sum over n, we get a group of all simple polytopes: $\mathbb{P} = \bigoplus_n \mathbb{P}_n$. This abelian group possesses the structure of a graded differential ring. The multiplication is defined on basis elements by the formula $P \cdot Q = P \times Q$ and extended by linearity on the whole group. The differentiation is defined on basis elements by $dP = \sum_{\mathcal{F} \subset P} \mathcal{F}$, which associates to a simple polytope a formal sum of its facets. Differentiation is extended by linearity to the whole group $\mathbb{P}$.

To each simple polytope $P \in \mathbb{P}_n$ corresponds a face polynomial $F(P)$ in two variables

$$F(P)(\alpha, t) = \sum_{F \subseteq P} \alpha^{\dim F} t^{n-\dim F}, \tag{7}$$

where sum is taken over all faces including P itself. This correspondence can be extended by linearity to the linear map $F \colon \mathbb{P} \to \mathbb{Z}[\alpha, t]$ to the ring of polynomials. Then F is a differential ring homomorphism, in particular, the commutation formula holds:

$$F(dP) = \frac{\partial}{\partial t} F(P) \tag{8}$$

This result is used to give a simple proof of Dehn-Sommerville relations by induction on n. The Dehn-Sommerville relations can be written in the following form: for any simple polytope P there holds

$$F(P)(1, t) = F(P)(-1, t+1). \tag{9}$$

The next statement shows how the general theory of K_P works for simple polytope P.

Proposition 4.11. *Let P be a simple polytope of dimension n.*

1)The polynomial $F_P(\alpha, t)$ coincides with a face polynomial $F(P)(\alpha, t)$ defined by 7.

2)Formula 8 follows from lemma 4.7 applied to the complex K_P.

3)Formula 9 follows from formula 4.10 applied to the complex K_P.

PROOF. We give a sketch of a proof. Technical details can be easily restored.

1)For a face F of a simple polytope P we have $m(F) = n - \dim F$ (that is a classical fact). Therefore $F_P(\alpha, t)$ is a homogeneous polynomial and coincides with the face polynomial $F(P)(\alpha, t)$.

2)Let $\mathcal{F}_i$ be a facet of P and i be a vertex of K_P corresponding to this facet. If P is simple, then $K_{\mathcal{F}_i} = \mathrm{link}_{K_P}\, i$.

The complex K_P is $(n-1)$-sphere in the case of simple polytope and by example 4.3 any simplex of K_P is a face simplex. Therefore $f_{K_P}(t) = \sum\limits_{\sigma \in K_P} t^{|\sigma|} = \sum\limits_{\sigma \in F(K_P)} t^{|\sigma|} = F_P(1, t) = F(P)(1, t)$.

If we forget about complementary variable α in formula 8 and use the definitions we get $\sum_{\mathcal{F}_i \subset P} F(\mathcal{F}_i)(1, t) = \frac{d}{dt} F(P)(1, t)$. Substituting $F(\mathcal{F})(1, t)$ by $f_{\mathrm{link}\, i}(t)$ and $F(P)(1, t)$ by $f_{K_P}(t)$, formula 8 reduces exactly to 4.7.

3)To prove the third part of the statement note that $f_{K_P}(t) = F_P(-1, t+1)$ by the corollary 4.10. On the other hand, $f_{K_P} = F_P(1, t)$ by the observation above. This gives Dehn-Sommerville relations $F(P)(1, t) = F(P)(-1, t+1)$. □

Proposition 4.11 serves as a link between theory of ring of polytopes and the general construction of K_P.

We now turn to the general situation when P is nonsimple.

Lemma 4.12. *Let P be a convex polytope. Then there is a coefficient-wise inequality $F_P(1, t) \leqslant F_P(-1, t+1)$, which turns to an equality for terms t^0 and t^1.*

PROOF. By definition, $F_P(1, t) = \sum\limits_{\sigma \in F(K_P)} t^{|\sigma|}$. By corollary 4.10, we have $F_P(-1, t+1) = \sum\limits_{\sigma \in K_P} t^{|\sigma|}$. But $\sum\limits_{\sigma \in F(K_P)} t^{|\sigma|} \leqslant \sum\limits_{\sigma \in K_P} t^{|\sigma|}$, since the second sum is taken over greater set.

Complex K_P is a reduced P-complex, therefore each of its vertices contributes in both sums $\sum\limits_{\sigma \in F(K_P)} t^{|\sigma|}$ and $\sum\limits_{\sigma \in K_P} t^{|\sigma|}$. Hence, the inequality for coefficients of $\sum\limits_{\sigma \in F(K_P)} t^{|\sigma|} \leqslant \sum\limits_{\sigma \in K_P} t^{|\sigma|}$ turns to an equality for the constant term and t. □

EXAMPLE 4.6. Let P be a 3-dimensional polytope. Consider the following numbers: m — number of facets, e — number of edges, a_i — number of vertices which are contained in exactly i facets, where $i \geqslant 3$. Any edge is contained in exactly two facets. Then by definition,

$$F_P(\alpha, t) = \alpha^3 + m\alpha^2 t + e\alpha t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \dots$$

Writing down the inequality of lemma 4.12, we get

$$1 + mt + et^2 + \sum_{i \geqslant 3} a_i t^i \leqslant -1 + m(t+1) - e(t+1)^2 + \sum_{i \geqslant 3} a_i (t+1)^i.$$

For the terms t^0 and t^1 there is an equality. For free term we have

$$1 = -1 + m - e + \sum_{i\geqslant 3} a_i,$$

which is just an Euler formula. For t^1 there holds:

$$m = m - 2e + \sum_{i\geqslant 3} ia_i;$$

$$2e = \sum_{i\geqslant 3} ia_i.$$

This can be seen directly from the edge graph of P.

EXAMPLE 4.7. Consider a polytope P of dimension 4. Let m be the number of its facets, r — the number of its ridges (that is faces of dimension 2), a_i — the number of its edges which are contained in exactly i facets, and b_i — the number of vertices of P which are contained in exactly i facets. Each ridge is an intersection of two facets. Thus

$$F_P(\alpha, t) = \alpha^4 + m\alpha^3 t + r\alpha^2 t^2 + \sum_{i\geqslant 3} a_i \alpha t^i + \sum_{i\geqslant 4} b_i t^i.$$

So the inequality of lemma 4.12 has the form

$$1 + mt + rt^2 + a_3 t^3 + \sum_{i\geqslant 4} (a_i + b_i) t^i \leqslant$$
$$\leqslant 1 - m(t+1) + r(t+1)^2 - a_3(t+1)^3 + \sum_{i\geqslant 4} (b_i - a_i)(t+1)^i.$$

For the free terms there holds an equality

$$1 = 1 - m + r - \sum a_i + \sum b_i.$$

This is the Euler-Poincare formula as was expected. For the t^1-terms the equality have the form

$$1 + m = 1 - m + 2r - \sum_{i\geqslant 3} ia_i + \sum_{i\geqslant 4} ib_i;$$

$$2m - 2r = \sum_{i\geqslant 4} ib_i - \sum_{i\geqslant 3} ia_i. \qquad (10)$$

The last equality can be rewritten in terms of flag f-numbers of a polytope P. Indeed, $\sum ia_i$ is exactly the number of flags $F \subset \mathcal{F}$, where F is an edge and $\mathcal{F}$ is a facet of P. So forth, $\sum ia_i = f_{\{1,3\}}$. Similarly, $\sum ib_i = f_{\{0,3\}}$ and $2e = f_{\{2,3\}}$. Therefore, relation 10 has the form

$$2f_{\{3\}} = f_{\{0,3\}} - f_{\{1,3\}} + f_{\{2,3\}}.$$

This is exactly one of the generalized Dehn-Sommerville relations, discovered by M. Bayer and L. Billera [**5**].

Lemma 4.13. *Let P be n-dimensional polytope, $f_{\{i\}}$ — the number of i-dimensional faces of P and $f_{\{i,n-1\}}$ — the number of pairs $F \subset \mathcal{F}$, where $\dim F = i$ and $\dim \mathcal{F} = n-1$. Then*

$$f_{\{n-1\}}(1+(-1)^n) = \sum_{i=0}^{n-2} (-1)^i f_{\{i,n-1\}}. \tag{11}$$

PROOF. Let $f_{i,k}$ be the number of i-dimensional faces of P which are contained in exactly k facets. Then by definition, $F_P(\alpha,t) = \sum_{i,k} f_{i,k}\alpha^i t^k$. In order to apply lemma 4.12 consider t^0 and t^1-terms of polynomials $F_P(1,t)$ and $F_P(-1,t+1)$. We have:

$$F_P(1,t) = A_0 + A_1 t + O(t^2),$$

where $A_0 = \sum_i f_{i,0} = 1$, $A_1 = \sum_i f_{i,1} = f_{\{n-1\}}$, since the only face which is not contained in any facet is P itself and the only faces which are contained in exactly 1 facets are facets. The symbol $O(t^2)$ stands for terms of degree greater than 1. Proceeding further, we get

$$F_P(-1,t+1) = \sum_{i,k} f_{i,k}(-1)^i (t+1)^k = B_0 + B_1 t + O(t^2),$$

where

$$B_0 = \sum_{i,k} f_{i,k}(-1)^i = \sum_i (-1)^i f_{\{i\}},$$

$$B_1 = \sum_{i,k} f_{i,k} k(-1)^i = \sum_i (-1)^i \sum_k f_{i,k} k = \sum_{i=0}^{n-2} (-1)^i f_{\{i,n-1\}} + (-1)^{n-1} f_{\{n-1\}}.$$

Therefore, by lemma 4.12 we get

$$A_0 = B_0,$$
$$1 = \sum_i (-1)^i f_{\{i\}},$$

which is Euler-Poincare relation for a polytope. We also have

$$A_1 = B_1,$$
$$f_{\{n-1\}} = \sum_{i=0}^{n-2} (-1)^i f_{\{i,n-1\}} + (-1)^{n-1} f_{\{n-1\}},$$

which is the required formula. □

REMARK. The equation 11 is exactly one of Bayer-Billera relations [**5**].

REMARK. The equalities of lemma 4.12 can be reduced to the relation on flag f-numbers. Nevertheless, the next example shows that coefficients of two-dimensional face polynomial $F_P(\alpha,t)$ can not be expressed in terms of flag f-numbers in general.

EXAMPLE 4.8. There exist two polytopes P and Q which have the same flag f-numbers, but $F_P(\alpha,t) \neq F_Q(\alpha,t)$. Such polytopes P and $Q = P^*$ are depicted in figure 5. Their flag numbers are the same: $f_{\{0\}} = 7, f_{\{1\}} = 12, f_{\{2\}} = 7, f_{\{0,1\}} = 2f_{\{1\}} = 24, f_{\{0,2\}} = f_{\{1,2\}} = 24$. But their two-dimensional face polynomials are different: $F_P(\alpha,t) = \alpha^3 + 7\alpha^2 t + 12\alpha t^2 + 4t^3 + 3t^4$, $F_{P^*}(\alpha,t) = \alpha^3 + 7\alpha^2 t + 12\alpha t^2 + 5t^3 + t^4 + t^5$.

5. Buchstaber number of a convex polytope

Recall that there are several definitions of the Buchstaber number for different objects.

Definition 5.1. *The Buchstaber number $s(P)$ of a convex polytope P is defined as a maximal rank of toric subgroups of T^m acting freely on $\mathcal{Z}_P \subset \mathbb{C}^m$.*

The definition 5.1 generalizes the definition of $s(P)$ for simple polytopes.

Definition 5.2. *Let G be an arbitrary hypergraph. Then a moment-angle complex $\mathcal{Z}_G$ is defined. The torus T^m acts on $\mathcal{Z}_G$ coordinatewise. The Buchstaber number $s(G)$ of a hypergraph is defined as a maximal rank of toric subgroups of T^m acting freely on $\mathcal{Z}_G$.*

The definition 5.2 generalizes the definition of $s(K)$ for simplicial complexes.

Definition 5.3. *Let U be a complement to some coordinate space arrangement. Then there is a coordinatewise action of torus on U. The Buchstaber number $s(U)$ of a coordinate space arrangement is a maximal rank of toric subgroups of T^m acting freely on U.*

Proposition 5.1. *For a general convex polytope P there holds:*

$$s(P) = s(G_P) = s(K_P) = s(U_P)$$

in the sense of definitions above.

Proof. Consider a general situation in which a torus T^m acts on a space X. Denote stabilizers of this action by $S_1, \dots, S_l$. A subgroup $G \subseteq T^m$ acts freely on X iff $G \cap S_i = \{\mathbf{1}\} \in T^m$ for any stabilizer S_i.

To prove that $s(P) = s(K_P) = s(U_P)$ we show that actions of T^m on $\mathcal{Z}_P$, $\mathcal{Z}_{K_P}$ and U_{K_P} have the same stabilizers. Indeed, in all three cases stabilizers are the subgroups $G = T^{\{i_1,\dots,i_k\}}$ where $\mathcal{F}_{i_1} \cap \dots \cap \mathcal{F}_{i_k} \neq \varnothing$. Such a subgroup stabilizes a point over $x \in \mathcal{F}_{i_1} \cap \dots \cap \mathcal{F}_{i_k}$ in case of $\mathcal{Z}_P$. The same subgroup stabilizes a point $(\varepsilon_1, \dots, \varepsilon_m) \in (D^2, S^1)^{K_P} = \mathcal{Z}_{K_P} \subseteq U_{K_P}$, where $\varepsilon_j = 0$ for $j \in \{i_1, \dots, i_k\}$ and 1 otherwise. Since the sets of stabilizers are the same in all three cases we have the same subgroups acting freely on corresponding spaces.

The equality $s(G_P) = s(K_P)$ follows from $\mathcal{Z}_{G_P} = \mathcal{Z}_{K_P}$. □

Example 5.1. We may define a pyramid over arbitrary convex polytope Q. The pyramid $\operatorname{pyr} Q$ is a convex hull of Q and a point v, that does not lie in a plane of Q. A pyramid is an operator on a set of all convex polytopes. Note that this operator does not make sense on the set of all simple polytopes, since $\operatorname{pyr} Q$ may be (and usually is) non-simple even if Q is simple.

There are many interesting properties of the operator pyr. For example, $(\operatorname{pyr} Q)^* = \operatorname{pyr} Q^*$. Therefore if $Q^* = Q$, then $(\operatorname{pyr} Q)^* = \operatorname{pyr} Q$. This gives, in particular, that a pyramid over a square on a figure 1 is self-polar since square is polar to itself.

Also we have $s(\operatorname{pyr} Q) = 1$ for any convex polytope Q. To prove this consider all facets $\mathcal{F}_1, \dots, \mathcal{F}_{m-1}$ of a $P = \operatorname{pyr} Q$ excluding its base. There m is the number of all facets of P. We have $\mathcal{F}_1 \cap \dots \cap \mathcal{F}_{m-1} \neq \varnothing$ since this intersection is a cone point. Therefore there is a stabilizer $T^{\{1,\dots,m-1\}} \subset T^m$ of the action of the torus on $\mathcal{Z}_{K_P}$. The rank of this stabilizer is $m-1$. Since any subgroup of T^m acting freely on $\mathcal{Z}_{K_P}$ should intersect stabilizer only in the unit, the rank of such subgroup should be less than or equal to 1. But for any polytope P

there exists a diagonal subgroup of a torus acting freely on $\mathcal{Z}_P$. Therefore $s(P)$ is exactly 1 in the case of pyramid.

We have a conjecture that pyramids are the only polytopes that have a Buchstaber number equal to 1. The similar fact for simple polytopes is true: the only simple polytopes that have a Buchstaber number 1 are simplices (which are pyramids). This was proved in [**11**].

Since the Buchstaber number of a polytope is the same as Buchstaber number of an appropriate simplicial complex we may write some estimations according to [**15**],[**11**],[**1**].

Proposition 5.2. *Let P and Q be polytopes with m_P and m_Q facets respectively. Then*

$$m-\gamma(P)\leqslant s(P)\leqslant m-\dim(K_P)-1,$$
$$s(P)\leqslant m-\lceil\log_2(\gamma(P)+1)\rceil,$$
$$s(P\times Q)\geqslant s(P)+s(Q),$$
$$s(P\times Q)\leqslant \min(s(P)+m_Q-\dim(K_Q),s(Q)+m_P-\dim(K_P)).$$

There $\gamma(P)$ is a minimal number of colors needed to paint the facets of P in such a way that intersecting facets have different colors.

Remark. For non-simple polytopes P there exist stabilizers $T^{\{i_1,\dots,i_k\}}$ of dimension $k>n$. Therefore $s(P)<m-n$.

6. Moment-angle complexes and Betti numbers

Let $\mathbb{Z}[K]$ denote the a Stanley-Reisner ring of a simplicial complex K [**19**],[**9**]. By [**6**] the isomorphism class of Stanley-Reisner ring defines the structure of K uniquely. For a polytope P define Stanley-Reisner ring $\mathbb{Z}[P]$ by $\mathbb{Z}[P]=\mathbb{Z}[K_P]$. Then the isomorphism class of the ring $\mathbb{Z}[P]$ defines the combinatorial class of P uniquely since P can be restored from K_P.

We recall the definition of Betti numbers for simplicial complexes. Details may be found in [**9**], section 3.

Definition 6.1. *Let K be a simplicial complex on m vertices. Set*

$$\beta^{-i,2j}(K)=\dim_{\mathbb{Z}}\operatorname{Tor}^{-i,2j}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K],\mathbb{Z}),\quad 0\leqslant i,j\leqslant m,$$

where

1) $\mathbb{Z}[K]$ is a Stanley-Reisner ring of a complex K. The natural projection $\mathbb{Z}[v_1,\dots,v_m]\to\mathbb{Z}[K]$ defines a structure of a $\mathbb{Z}[v_1,\dots,v_m]$-module on $\mathbb{Z}[K]$.

2) A natural augmentation $\mathbb{Z}[v_1,\dots,v_m]\to\mathbb{Z}$ defines a structure of $\mathbb{Z}[v_1,\dots,v_m]$-module on $\mathbb{Z}$.

3) The grading $-i$ corresponds to the term of resolution, and the grading $2j$ corresponds to the natural grading in the ring of polynomials.

Then numbers $\beta^{-i,2j}(K)$ are called Betti numbers of a simplicial complex K.

We have an isomorphism of graded algebras $H^*(\mathcal{Z}_K,\mathbb{Z})\cong\operatorname{Tor}^*_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K],\mathbb{Z})$, where the grading of the Tor-algebra is given by total degree. This isomorphism can also be used to define a bigraded structure on $H^*(\mathcal{Z}_K,\mathbb{Z})$.

Definition 6.2. *For a polytope P define Betti numbers by $\beta^{-i,2j}(P)=\beta^{-i,2j}(K_P)$.*

Hochster's formula can be written for general convex polytopes in its usual form.

Proposition 6.1 (Hochster's theorem for convex polytopes)**.** *Let $F_1, \dots, F_m$ be a set of facets of a polytope P. Then*

$$\beta^{-i,2j}(P) = \sum_{\omega \subseteq [m], |\omega|=j} \dim_{\mathbb{Z}} \tilde{H}^{j-i-1}(F_\omega; \mathbb{Z}),$$

where $F_\omega = \bigcup\limits_{i \in \omega} F_i \subseteq P$.

Proof. By the definition $\beta^{-i,2j}(P) = \beta^{-i,2j}(K_P) = \sum\limits_{\omega \subseteq [m], |\omega|=j} \dim_{\mathbb{Z}} \tilde{H}^{j-i-1}(K_\omega; \mathbb{Z})$, where K_w is a complete subcomplex of K_P spanned on ω. Second equality represents Hochster's formula for simplicial complexes. We claim that $K_\omega \simeq F_\omega$. Indeed, there is a contractible cover of $F_\omega = \bigcup\limits_{i \in \omega} F_i \subseteq P$ by subsets F_i. One can see, that the nerve of this cover is K_ω. Therefore $K_\omega \simeq F_\omega$ and $\tilde{H}^{j-i-1}(K_\omega; \mathbb{Z}) \cong \tilde{H}^{j-i-1}(F_\omega; \mathbb{Z})$ which concludes the proof. □

We have, by definition, $\dim H^p(\mathcal{Z}_{K_P}, \mathbb{Z}) = \sum\limits_{-i+2j=p} \beta^{-i,2j}(P)$. The next statement shows that we can substitute $\mathcal{Z}_{K_P}$ by $\mathcal{Z}_P$ in this formula.

Theorem 6.2. *For any convex polytope P there holds*

$$\mathcal{Z}_{K_P} \simeq \mathcal{Z}_P.$$

Remark. For non-simple P the spaces $\mathcal{Z}_P$ and $\mathcal{Z}_{K_P}$ are not homeomorphic. This can be seen by dimensional reasons. If a polytope P has dimension n and is not simple, then K_P contains a simplex τ such that $|\tau| > n$. Therefore $\dim \mathcal{Z}_{K_P} = \dim (D^2, S^1)^K > m + n$. But $\dim \mathcal{Z}_P = m + n$ since $\mathcal{Z}_P$ can be described as an identification space of $P \times T^m$ (see the proof below).

Proof. To prove this we use properties of homotopy colimits. First of all recall that a space $\mathcal{Z}_{K_P}$ can be described as a usual colimit of a diagram. Let $\mathrm{cat}(K_P)$ be a small category associated to simplicial complex K_P. The objects of $\mathrm{cat}(K_P)$ are simplices of K_P. The morphisms of $\mathrm{cat}(K_P)$ are inclusions of simplices. This means that there exists exactly one morphism from σ to τ whenever $\sigma \subseteq \tau$. In such terms the empty simplex is the initial object of $\mathrm{cat}(K_P)$.

The diagram $\bar{D}_{K_P} \colon \mathrm{cat}(K_P) \to \mathrm{Top}$ is defined by $\bar{D}_{K_P}(\sigma) = (D^2)^\sigma \times T^{[m] \setminus \sigma}$ and $\bar{D}_{K_P}(\sigma \hookrightarrow \tau) = ((D^2)^\sigma \times T^{[m] \setminus \sigma} \hookrightarrow (D^2)^\tau \times T^{[m] \setminus \tau})$. Then $\mathrm{colim}\, \bar{D}_{K_P} \cong \mathcal{Z}_{K_P}$ (see details in [**17**]).

Further we will use a description of $\mathcal{Z}_P$ using colimit.

1) The first step is to realize $\mathcal{Z}_P$ as a quotient space $\mathcal{Z}_P \cong P \times T^m / \sim$, where $(x_1, t_1) \sim (x_2, t_2)$ whenever $x_1 = x_2$ and $t_1 t_2^{-1} \in T^{\tilde{\sigma}(x)}$. The function $\tilde{\sigma}$ taking values in $2^{[m]}$ was introduced in section 2. Its value in point x is the set of all facets containing x.

To prove that the quotient space is homeomorphic to $\mathcal{Z}_P$ consider an inclusion map $i_P = i \circ j_P \colon P \to \mathbb{R}^m \hookrightarrow \mathbb{C}^m$ which is a section of the map $p \colon \mathcal{Z}_P \to P$. Then a map $h \colon P \times T^m \to \mathcal{Z}_P$, $h(p,t) = t \cdot i_P(x)$ induces a required homeomorphism from $\mathcal{Z}_P \cong P \times T^m / \sim$.

2) We now realize a quotient space as a homotopy colimit.

Consider a small category $\mathrm{cat}(P)$. Its objects are all faces of a polytope P (here we consider the whole polytope as a face of itself). The morphisms are given by inverse inclusions. This means that there exists exactly one morphism from F to G whenever $F \supseteq G$. The function $\tilde{\sigma}$ can be treated as a functor between small categories $\tilde{\sigma}\colon \mathrm{cat}(P) \to \mathrm{cat}(K_P)$. Indeed, any face F defines a hyperedge in G_P thus a simplex in K_P. When $F \supseteq G$ we have $\tilde{\sigma}(F) \subseteq \tilde{\sigma}(G)$.

Define a functor $D_P\colon \mathrm{cat}(P) \to \mathrm{Top}$ by the rules $D_P(F) = T^{[m]\setminus\tilde{\sigma}(F)}$ and $D_P(F \supseteq G) = q_{F,G}\colon T^{[m]\setminus\tilde{\sigma}(F)} \to T^{[m]\setminus\tilde{\sigma}(F)}/T^{\tilde{\sigma}(G)\setminus\tilde{\sigma}(F)} \cong T^{[m]\setminus\tilde{\sigma}(G)}$ — a natural projection to the quotient group.

We claim that a homotopy colimit of a diagram D_P is exactly the space $P \times T^m/\sim$. By definition, the homotopy colimit $\operatorname{hocolim} D_P$ is a quotient of $\coprod\limits_{F\in\mathrm{Ob}\,\mathrm{cat}(P)} |F \downarrow \mathrm{cat}(P)| \times D_P(F)$ under identifications $(x, D_P(F \supseteq G)(y)) \sim (x, y)$ for $x \in |G \downarrow \mathrm{cat}(P)| \subseteq |F \downarrow \mathrm{cat}(P)|$ and $y \in D_P(F)$. In our case the geometrical realization of an undercategory $|F \downarrow \mathrm{cat}(P)|$ is a barycentric subdivision of F itself. Moreover inclusions of undercategories coincide with inclusions of faces in P. Thus $\operatorname{hocolim} D_P$ is a quotient of $\coprod\limits_{F\in\mathrm{Ob}\,\mathrm{cat}(P)} F \times T^{[m]\setminus\tilde{\sigma}(F)}$ under identifications $(x, t) \sim (x, q_{F,G}(t))$ for $x \in G \subseteq F$ and $t \in T^{\tilde{\sigma}(F)}$. Such a space is obviously homeomorphic to $P \times T^m/\sim$ introduced earlier.

3) We already have all ingredients to complete the proof of the theorem.

Consider an ancillary diagram $\bar{D}_P\colon \mathrm{cat}(P) \to \mathrm{Top}$. Let $\bar{D}_P(F) = T^{[m]\setminus\tilde{\sigma}(F)} \times (D^2)^{\tilde{\sigma(F)}}$ and $\bar{D}(F \supseteq G)$ is the inclusion of $T^{[m]\setminus\tilde{\sigma}(F)} \times (D^2)^{\tilde{\sigma}(F)}$ into $T^{[m]\setminus\tilde{\sigma}(G)} \times (D^2)^{\tilde{\sigma}(G)}$. All the maps in the diagram $\bar{D}_P$ are homotopically equivalent to the corresponding maps in the diagram D_P. Therefore we have

$$\operatorname{hocolim} D_P \simeq \operatorname{hocolim} \bar{D}_P \simeq \operatorname{colim} \bar{D}_P.$$

The second equivalence holds since all the maps in $\bar{D}_P$ are cofibrations.

Note that $\bar{D}_P = D_{K_P} \circ \tilde{\sigma}\colon \mathrm{cat}(P) \to \mathrm{Top}$. So $\bar{D}_P$ may be treated as a subdiagram of $\mathrm{cat} K_P$. It can be directly shown that these two diagrams have the same colimit equal to $\bigcup\limits_{F\in P} (D^2)^{\tilde{\sigma}(F)} \times T^{m\setminus\tilde{\sigma}(F)} = (D^2, S^1)^{K_P}$. This concludes the proof.

$\square$

The theorem states that there is an additional homotopy equivalence $\mathcal{Z}_P \simeq \mathcal{Z}_{K_P}$ in a scheme 1. This theorem together with proposition 5.1 lets us think that a complex K_P is a nice substitute for a convex polytope P.

Example 6.1. Consider a pyramid P over a square (figure 1). As a consequence of a proof of theorem 6.2 we have $\mathcal{Z}_P \cong (P \times T^5)/\sim$. Restricting this construction to the first facet $\mathcal{F}_1 = \triangle ABC$ (which is one of side facets), we get the space $p^{-1}(\mathcal{F}_1) \cong \mathcal{F}_1 \times T^4/\sim$ where T^4 stands for the product of circles corresponding to facets $\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5$. The equivalence relation in this formula is given by

$$
(x,t) \sim (y,s) \Leftrightarrow \begin{cases} x = y \in (AB), ts^{-1} \in T^{\{5\}}, \\ x = y \in (BC), ts^{-1} \in T^{\{2\}}, \\ x = y \in (AC), ts^{-1} \in T^{\{4\}}, \\ x = y = A, ts^{-1} \in T^{\{4,5\}}, \\ x = y = B, ts^{-1} \in T^{\{2,5\}}, \\ x = y = C, ts^{-1} \in T^{\{2,3,4\}}. \end{cases}
$$

We see that the restriction of the moment-angle map p over a facet $\mathcal{F}_1$ does not coincide with $\mathcal{Z}_{F_1} \times T^1$. It can be deduced from the fact that $p^{-1}(\mathcal{F}_1)$ is simply connected: all basic cycles of a torus can be contracted using the equivalence relation. But $\mathcal{Z}_{F_1} \times T^1$ is not simply connected. This example shows that a restriction of a moment-angle map over the facet is not so well behaved as in the case of simple polytopes.

We hope that invariants of P defined via the complex K_P will help to solve some problems. For example, the problem 1 may lead to better understanding of self-polar polytopes.

Instead of dealing with cohomology ring $H^*(\mathcal{Z}_{K_P})$ we can look only at Betti numbers of a complex K_P.

Example 6.2. The only simple polytopes which are combinatorially equivalent to their polar polytopes are simplices Δ^n in each dimension and m-gons P_m in dimension 2. We have $s(\Delta^n) = 1$ and $s(P_m) = m - 2$.

Example 6.3. A pyramid P shown on the figure 1 is equivalent to its polar. For such pyramid we have $s(P) = 1$ and nonzero Betti numbers of K_P are $\beta^{0,0}(P) = 1$, $\beta^{-1,6}(P) = 2$, $\beta^{-2,10}(P) = 1$.

There is a nice way to calculate Betti numbers of pyramids using Betti numbers of their bases.

Proposition 6.3. *For any convex polytope Q there holds $\beta^{-i,2j}(\operatorname{pyr} Q) = \beta^{-i,2(j-1)}(Q)$ for $j > 0$.*

Remark. The only nonzero Betti number with $j = 0$ is $\beta^{0,0}(K) = 1$ for any simplicial complex K.

Proof. We use a Hochster formula 6.1 for a polytope $\operatorname{pyr} Q$:

$$
\beta^{-i,2j}(\operatorname{pyr} Q) = \sum_{\omega \subseteq [m], |\omega| = j} \dim_{\mathbb{Z}} \tilde{H}^{j-i-1}(F_\omega; \mathbb{Z}),
$$

where $F_\omega = \bigcup_{i \in \omega} \mathcal{F}_i \subseteq \operatorname{pyr} Q$. We suppose that a base Q of the pyramid is the first facet in a given enumeration, $Q = \mathcal{F}_1$. If the set ω does not contain 1, then the set F_ω is contractible to the apex of the pyramid. Therefore such terms do not contribute at the sum at the right.

Suppose $\omega = \{1, i_2, \ldots, i_j\}$. Any facet $\mathcal{F}_i$ except Q is a pyramid over a facet $\tilde{\mathcal{F}}_i$ of a polytope Q. Contracting Q in F_ω, we get $F_\omega \simeq \Sigma(\tilde{\mathcal{F}}_{i_2} \cup \cdots \cup \tilde{\mathcal{F}}_{i_j})$. The base of a suspension

is a space $\tilde{F}_{\omega\setminus\{1\}}$ for a polytope Q. Then by suspension isomorphism in cohomology we have

$$\sum_{\omega\subseteq[m],|\omega|=j} \dim_{\mathbb{Z}} \tilde{H}^{j-i-1}(F_\omega;\mathbb{Z}) = \sum_{\omega\subseteq[m]\setminus\{1\},|\omega|=j-1} \dim_{\mathbb{Z}} \tilde{H}^{j-i-1}(\Sigma\tilde{F}_{\omega\setminus\{1\}};\mathbb{Z}) =$$
$$= \sum_{\omega\subseteq[m]\setminus\{1\},|\omega|=j-1} \dim_{\mathbb{Z}} \tilde{H}^{j-i-2}(\tilde{F}_{\omega\setminus\{1\}};\mathbb{Z}) = \beta^{-i,2(j-1)}(Q). \quad (12)$$

This concludes the proof. □

Another statement is a trivial consequence of the multiplicativity of Betti numbers for simplicial complexes and proposition 3.2.

Proposition 6.4. *For convex polytopes P and Q there holds an equality*

$$\beta^{-i,2j}(P\times Q) = \sum_{\substack{i_1+i_2=i\\ j_1+j_2=j}} \beta^{-i_1,2j_1}(P)\beta^{-i_2,2j_2}(Q).$$

Using Betti numbers and Buchstaber number, we can find an obstruction for a polytope to be polar to itself. If $P = P^*$, then first of all $f_i(P) = f_{n-i}(P)$, where $f_i(P)$ is the number of i-faces of P. This means that f-vector $(f_0,\dots,f_n)$ is symmetric, which provides a necessary condition for a polytope to be self-polar. We also have $K_P = K_{P^*}$ therefore $\beta^{-i,2j}(P) = \beta^{-i,2j}(P^*)$ and $s(P) = s(P^*)$.

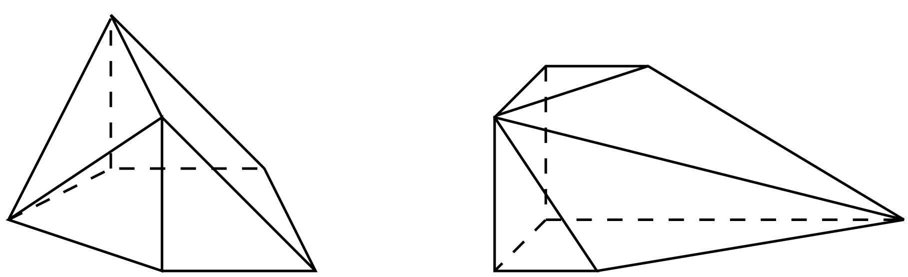

Figure 5. Polytope P and its polar P^*

Example 6.4. Consider two polytopes P and P^* depicted on figure 5. One can see that they are not equivalent (this was discussed in example 4.8). But we cannot distinguish these polytopes using only their f-vectors: they both are equal to $(7,12,7)$. Nevertheless, they have different Betti numbers. Nonzero Betti numbers of P are $\beta^{0,0}(P)=1$, $\beta^{-1,4}(P)=3$, $\beta^{-1,6}(P)=6$, $\beta^{-2,8}(P)=14$, $\beta^{-3,10}(P)=9$, $\beta^{-4,14}(P)=1$. Nonzero Betti numbers of P^* are $\beta^{0,0}(P^*)=1$, $\beta^{-1,4}(P^*)=2$, $\beta^{-1,6}(P^*)=6$, $\beta^{-2,8}(P^*)=15$, $\beta^{-3,10}(P^*)=9$, $\beta^{-4,14}(P^*)=1$. This example shows that Betti numbers are more strong invariants than f-vector. The computation was made using Macaulay2 software.

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